Automated Synthesis of Target-Dependent Programs for Polynomial Evaluation in Fixed-Point Arithmetic

Christophe Mouilleron    Amine Najahi    Guillaume Revy

Univ. Perpignan Via Domitia, DALI project-team
Univ. Montpellier 2, LIRMM, UMR 5506
CNRS, LIRMM, UMR 5506
Summary

Context and objectives

- Automated synthesis of fixed-point programs
  - work done within the french ANR DEFIS project ([http://defis.lip6.fr](http://defis.lip6.fr))
  - targeting critical systems

- Optimized code
  - use of the advanced instructions available on the target embedded systems

- Certified accuracy bounds using analytic approaches
  - contrarily to simulation based approaches

Achievements

1. A new arithmetic model for the synthesis of signed fixed-point programs
2. A modular approach to optimize the generated code based on instruction selection
   - the generated code can be optimized for accuracy, for latency, or for both
   - more than 10% of speedup and almost up to 1 bit of precision gained
Motivation

- In this talk, we will focus on polynomial evaluation
  - it frequently appears as a building block of some mathematical operator implementation \( \leadsto \) floating-point support emulation
  - it can be used to convert calls to floating-point operators into fixed-point code \( \leadsto \) fixed-point conversion

- **Remark:** There is a huge number of schemes to evaluate a given polynomial, even for small degree
  - degree-5 univariate polynomial \( \leadsto \) 2334244 different schemes

There is a need for the automation of the design of polynomial evaluation codes \( \leadsto \) CGPE.
Outline of the talk

1. The CGPE tool

2. Code optimization through instruction selection

3. Conclusion and perspectives
Outline of the talk

1. The CGPE tool

2. Code optimization through instruction selection

3. Conclusion and perspectives
Overview of CGPE

Goal of CGPE: automate the design of fast and certified C codes for evaluating univariate or bivariate polynomials in fixed-point arithmetic

- by using unsigned fixed-point arithmetic only
- by using the target architecture features (as much as possible)

Remarks on CGPE

- fast rightarrow that reduce the evaluation latency on a given target
- certified rightarrow for which we can bound the error entailed by the evaluation within the given target’s arithmetic
Global architecture of CGPE

- Architecture of CGPE ≈ architecture of a compiler
  - it proceeds in three main steps

1. Computation step ~ front-end
   - computes schemes reducing the evaluation latency on unbounded parallelism ~ DAG
   - considers only the cost of $\oplus$ and $\otimes$
Global architecture of CGPE

- Architecture of CGPE $\approx$ architecture of a compiler
  - it proceeds in three main steps

1. Computation step $\rightsqarrow$ front-end
   - computes schemes reducing the evaluation latency on unbounded parallelism $\rightsqarrow$ DAG
   - considers only the cost of $\oplus$ and $\otimes$

2. Filtering step $\rightsqarrow$ middle-end
   - prunes the DAGs that do not satisfy different criteria:
     - latency $\rightsqarrow$ scheduling filter,
     - accuracy $\rightsqarrow$ numerical filter, ...

---

A. Najahi (DALI UPVD/LIRMM, UM2, CNRS)
Automated Synthesis of Target-Dependent Programs for Polynomial Evaluation in Fixed-Point Arithmetic
7/20
Global architecture of CGPE

- Architecture of CGPE ≈ architecture of a compiler
  - it proceeds in three main steps

1. Computation step  \(\rightsquigarrow\) front-end
   - computes schemes reducing the evaluation latency on unbounded parallelism  \(\rightsquigarrow\) DAG
   - considers only the cost of \(\oplus\) and \(\otimes\)

2. Filtering step  \(\rightsquigarrow\) middle-end
   - prunes the DAGs that do not satisfy different criteria:
     - latency  \(\rightsquigarrow\) scheduling filter,
     - accuracy  \(\rightsquigarrow\) numerical filter, ...

3. Generation step  \(\rightsquigarrow\) back-end
   - generates C codes and Gappa accuracy certificates
Recent contributions to CGPE

- no support for signed fixed-point arithmetic
  - handling of variables of constants sign
    - problem: CGPE fails in evaluating polynomials around one of its roots

- hypotheses are made on the format of the inputs
  - no shift operators are allowed during the evaluation
    - problem: CGPE fails in evaluating polynomials with inputs having incorrect formats

- simple description of the target architecture
  - no handling of advanced operators
    - problem: CGPE fails in making the most out of any advanced instructions
Recent contributions to CGPE

- no support for signed fixed-point arithmetic
  - handling of variables of constants sign
  - problem: CGPE fails in evaluating polynomials around one of its roots

- hypotheses are made on the format of the inputs
  - no shift operators are allowed during the evaluation
  - problem: CGPE fails in evaluating polynomials with inputs having incorrect formats

- simple description of the target architecture
  - no handling of advanced operators
  - problem: CGPE fails in making the most out of any advanced instructions
Recent contributions to CGPE

- no support for signed fixed-point arithmetic  
  - handling of variables of constants sign  
  - problem: CGPE fails in evaluating polynomials around one of its roots

- hypotheses are made on the format of the inputs  
  - no shift operators are allowed during the evaluation  
  - problem: CGPE fails in evaluating polynomials with inputs having incorrect formats

- simple description of the target architecture  
  - no handling of advanced operators  
  - problem: CGPE fails in making the most out of any advanced instructions  
  - main motivation: it may absorb shifts appearing in the DAG, eventually in the critical path
Outline of the talk

1. The CGPE tool

2. Code optimization through instruction selection

3. Conclusion and perspectives
Introduction to instruction selection

- It is a well known problem in compilation proven to be NP-complete on DAGs.

- Usually solved using a tiling algorithm:
  - **input:**
    - a DAG representing an arithmetic expression,
    - a set of tiles, with a cost for each,
    - a function that associates a cost to a DAG.
  - **output:** a set of covering tiles that minimize the cost function.

- Examples of advanced instructions
  - **fma on IEEE processors** → $a \times b + c$ with only one final rounding
  - **mulacc on some DSP** → $a \times b + c$
  - **shift-and-add instruction on the ST231** → $a \ll b + c$ in 1 cycle, with $b \in \{1, \ldots, 4\}$
Motivation of using instruction selection inside CGPE

- **Related work:** Voronenko and Püschel from the Spiral group
  - Automatic Generation of Implementations for DSP Transforms on Fused Multiply-Add Architectures (2004)
  - Mechanical Derivation of Fused Multiply-Add Algorithms for Linear Transforms (2007)
Motivation of using instruction selection inside CGPE

- **Related work:** Voronenko and Püschel from the Spiral group
  - Automatic Generation of Implementations for DSP Transforms on Fused Multiply-Add Architectures (2004)
  - Mechanical Derivation of Fused Multiply-Add Algorithms for Linear Transforms (2007)

✓ they provide a short proof of optimality in the case of trees
✗ their method handles fma in DAGs but is not generic
Motivation of using instruction selection inside CGPE

- **Related work:** Voronenko and Püschel from the Spiral group
  - Automatic Generation of Implementations for DSP Transforms on Fused Multiply-Add Architectures (2004)
  - Mechanical Derivation of Fused Multiply-Add Algorithms for Linear Transforms (2007)

  ✔ they provide a short proof of optimality in the case of trees
  ❌ their method handles \texttt{fma} in DAGs but is not generic

- **Our goal is twofold:**
  1. to handle any advanced instruction described in an external XML file
  2. to integrate a numerical verification step in the process of instruction selection
For each instruction, the XML architecture description file contains:

- the name, the type (signed or unsigned), the latency (# cycles),
- a description of the pattern matched by the instruction,
- a C macro for emulating the instruction in software,
- and a piece of Gappa script for computing the error entailed by the instruction evaluation in fixed-point arithmetic.
The NOLTIS tiling algorithm

Near-Optimal Instruction Selection algorithm (Koes and Goldstein in CGO-2008)

1: BottomUpDP() + TopDownSelect()
2: ImproveCSEDecision()
3: BottomUpDP() + TopDownSelect()

Example: how to evaluate $a_0 + \left( (a_1 \cdot x) + \left( (a_2 \cdot (x \cdot x)) \ll 1 \right) \right)$?

- addition / shift $\rightsquigarrow$ 1 cycle
- shift-and-add $\rightsquigarrow$ 1 cycle
- multiplication $\rightsquigarrow$ 3 cycles
The NOLTIS tiling algorithm

Near-Optimal Instruction Selection algorithm (Koes and Goldstein in CGO-2008)

1: BottomUpDP() + TopDownSelect()
2: ImproveCSEDecision()
3: BottomUpDP() + TopDownSelect()

Example: how to evaluate $a_0 + \left( (a_1 \cdot x) + \left( (a_2 \cdot (x \cdot x)) \ll 1 \right) \right)$?

- Addition / shift $\sim 1$ cycle
- Shift-and-add $\sim 1$ cycle
- Multiplication $\sim 3$ cycles
The NOLTIS tiling algorithm

Near-Optimal Instruction Selection algorithm (Koes and Goldstein in CGO-2008)

1: BottomUpDP() + TopDownSelect()
2: ImproveCSEDecision()
3: BottomUpDP() + TopDownSelect()

Example: how to evaluate $a_0 + \left( (a_1 \cdot x) + ((a_2 \cdot (x \cdot x)) \ll 1) \right)$?

```
addition / shift $\sim 1$ cycle
shift-and-add $\sim 1$ cycle
multiplication $\sim 3$ cycles
```

BottomUpDP()
The NOLTIS tiling algorithm

Near-Optimal Instruction Selection algorithm (Koes and Goldstein in CGO-2008)

1: BottomUpDP() + TopDownSelect()
2: ImproveCSEDecision()
3: BottomUpDP() + TopDownSelect()

Example: how to evaluate $a_0 + \left( (a_1 \cdot x) + ( (a_2 \cdot (x \cdot x)) \ll 1 \right)$?

- Addition / shift $\leadsto$ 1 cycle
- Shift-and-add $\leadsto$ 1 cycle
- Multiplication $\leadsto$ 3 cycles
The NOLTIS tiling algorithm

Near-Optimal Instruction Selection algorithm (Koes and Goldstein in CGO-2008)

1: BottomUpDP() + TopDownSelect()
2: ImproveCSEDecision()
3: BottomUpDP() + TopDownSelect()

Example: how to evaluate $a_0 + \left( (a_1 \cdot x) + (a_2 \cdot (x \cdot x)) \ll 1 \right)$?

- Addition / shift $\sim 1$ cycle
- Shift-and-add $\sim 1$ cycle
- Multiplication $\sim 3$ cycles
The NOLTIS tiling algorithm

Near-Optimal Instruction Selection algorithm (Koes and Goldstein in CGO-2008)

1: BottomUpDP() + TopDownSelect()
2: ImproveCSEDecision()
3: BottomUpDP() + TopDownSelect()

Example: how to evaluate $a_0 + \left( (a_1 \cdot x) + (a_2 \cdot (x \cdot x)) \ll 1 \right)$?

Addition / shift $\leadsto$ 1 cycle
Shift-and-add $\leadsto$ 1 cycle
Multiplication $\leadsto$ 3 cycles

BottomUpDP()
The NOLTIS tiling algorithm

Near-Optimal Instruction Selection algorithm (Koes and Goldstein in CGO-2008)

1: BottomUpDP() + TopDownSelect()
2: ImproveCSEDecision()
3: BottomUpDP() + TopDownSelect()

Example: how to evaluate $a_0 + (a_1 \cdot x) + ((a_2 \cdot (x \cdot x)) \ll 1)$?

![Diagram showing the evaluation process with nodes representing operations and cycles]

- Addition / shift $\sim 1$ cycle
- Shift-and-add $\sim 1$ cycle
- Multiplication $\sim 3$ cycles
The NOLTIS tiling algorithm

Near-Optimal Instruction Selection algorithm (Koes and Goldstein in CGO-2008)

1: BottomUpDP() + TopDownSelect()
2: ImproveCSEDecision()
3: BottomUpDP() + TopDownSelect()

Example: how to evaluate $a_0 + \left( (a_1 \cdot x) + \left( (a_2 \cdot (x \cdot x)) \ll 1 \right) \right)$?

- Addition / shift $\sim 1$ cycle
- Shift-and-add $\sim 1$ cycle
- Multiplication $\sim 3$ cycles

TopDownSelect()
The NOLTIS tiling algorithm

Near-Optimal Instruction Selection algorithm (Koes and Goldstein in CGO-2008)

1: BottomUpDP() + TopDownSelect()
2: ImproveCSEDecision()
3: BottomUpDP() + TopDownSelect()

- Example: how to evaluate $a_0 + \left( (a_1 \cdot x) + ((a_2 \cdot (x \cdot x)) \ll 1) \right)$?

- In our case, only the first step of NOLTIS is valuable.

- NOLTIS algorithm mainly relies on the evaluation of a cost function. We have implemented three different cost functions:
  - number of operator (regardless commun subexpressions)
  - evaluation latency on unbounded parallelism
  - evaluation accuracy, computed by using the piece of Gappa script for each instruction
Remarks on instruction selection in CGPE

- A separation is achieved between the computation of the intermediate representation and the code generation process
  - we can generate codes according different criteria
  - we can generate target-dependent codes without writing new computation algorithms each time a new instruction is available
  - this general approach allows to tackle other problems (sum, dot-product, ...)

We are not bounded to basic instructions
- we can add many others advanced instructions or basic blocks
- this general approach allows to give some feedback on the eventual need of some new instructions
Remarks on instruction selection in CGPE

- A separation is achieved between the computation of the intermediate representation and the code generation process
  - we can generate codes according different criteria
  - we can generate target-dependent codes without writing new computation algorithms each time a new instruction is available
  - this general approach allows to tackle other problems (sum, dot-product, ...)

- We are not bounded to basic instructions
  - we can add many others advanced instructions or basic blocks
  - this general approach allows to give some feedback on the eventual need of some new instructions
Impact on the number of instructions

Figure: Average number of instructions in 50 synthesized codes, for the evaluation of polynomials of degree 5 up to 12 for various elementary functions.

- **Remark 1**: average reduction of 8.7 % up to 13.75 %

- **Remark 2**: interest of ST231 shift-and-add for \( \sin(x) \) implementation
  \( \leadsto \) reduction of 8.7 %

- **Remark 3**: interest of shift-and-add with right shift for \( \cos(x) \) and \( \log_2(1 + x) \) implementation
  \( \leadsto \) reduction of 12.8 % and 13.75 %, respectively
## Impact on the accuracy of some functions

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$I$</th>
<th>$d$</th>
<th>Accuracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\exp(x) - 1$</td>
<td>$[-0.25, 0.25]$</td>
<td>7</td>
<td>$-26.98$</td>
</tr>
<tr>
<td>$\exp(x)$</td>
<td>$[0, 1]$</td>
<td>7</td>
<td>$-13.94$</td>
</tr>
<tr>
<td>$\sin(x)$</td>
<td>$[-0.5, 0.5]$</td>
<td>9</td>
<td>$-18.95$</td>
</tr>
<tr>
<td>$\cos(x)$</td>
<td>$[-0.5, 0.25]$</td>
<td>5</td>
<td>$-27.01$</td>
</tr>
<tr>
<td>$\tan(x)$</td>
<td>$[0.25, 0.5]$</td>
<td>9</td>
<td>$-18.81$</td>
</tr>
<tr>
<td>$\log_2(1 + x)/x$</td>
<td>$[2^{-23}, 1]$</td>
<td>7</td>
<td>$-13.94$</td>
</tr>
<tr>
<td>$\sqrt{1 + x}$</td>
<td>$[2^{-23}, 1]$</td>
<td>7</td>
<td>$-13.94$</td>
</tr>
</tbody>
</table>

**Table:** Impact of the accuracy based selection step on the certified accuracy of the generated code for various functions.

- **Remark 1:** with a `mulacc` that computes $(a \ast b) + (c >> n)$ with $n \in \{1, \ldots, 31\}$ with one final rounding

- **Remark 2:** a gain of precision is obtained in all the cases, almost up to 1 bit of accuracy for $\exp, \sin, \log_2$ and $\sqrt{1 + x}$
Impact on the latency

- **Polynomial**: degree-7 polynomial approximating the function $\cos(x)$ over $[0, 2]$

- **Architecture**:
  - 1 cycle addition/subtraction and shift-and-add
  - 3-cycle multiplication and `mulacc`

<table>
<thead>
<tr>
<th></th>
<th>Without tiling</th>
<th>With tiling</th>
<th>Speed-up</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horner’s rule</td>
<td>41</td>
<td>34</td>
<td>$\approx 17.07%$</td>
</tr>
<tr>
<td>Estrin’s rule</td>
<td>16</td>
<td>14</td>
<td>$\approx 12.5%$</td>
</tr>
<tr>
<td>Best scheme</td>
<td>15</td>
<td>13</td>
<td>$\approx 13.33%$</td>
</tr>
</tbody>
</table>

**Table**: Latency in # cycles on unbounded parallelism, for various schemes, with and without tiling.
Outline of the talk

1. The CGPE tool

2. Code optimization through instruction selection

3. Conclusion and perspectives
Conclusion and perspectives

- Target-dependent code generation for fast and certified polynomial evaluation
  - in signed and unsigned fixed point arithmetic
  - using filter based on instruction selection, so as to make the most out of advanced instructions
  - selection according to different criteria: operator count, latency on unbounded parallelism, accuracy

- Further extensions of CGPE
  - this work has already been extended to sums and dot-products
    http://cgpe.gforge.inria.fr/
  - it has been used in higher level tools to generate fixed-point code for linear algebra programs FPLA
    http://perso.univ-perp.fr/mohamedamine.najahi/fpla/
  - to handle other arithmetics like the floating-point arithmetic, where the fma instruction is more and more ubiquitous
Automated Synthesis of Target-Dependent Programs for Polynomial Evaluation in Fixed-Point Arithmetic

Christophe Mouilleron   Amine Najahi   Guillaume Revy

Univ. Perpignan Via Domitia, DALI project-team
Univ. Montpellier 2, LIRMM, UMR 5506
CNRS, LIRMM, UMR 5506